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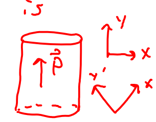
Vectors

~~Spacetime~~ Hitherto just "vectors"

Spacetime vectors correspond to real physical quantities with magnitudes (that may or may not be related to spacetime, e.g. $\Delta\vec{x}$, $\vec{E}$) and directions that do correspond to directions in spacetime.

They have 3 important features:

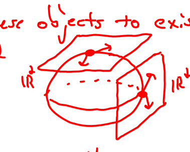
1. The vector itself is invariant under coordinate changes. Sometimes this is hard to visualize so consider a momentum vector on an infinite cylinder:



The vector will have different components in each coordinate system, but the actual direction does not change. If it did it could go from an ∞-direction to a periodic one!

2. The components of the vector should have a well-defined transformation under any change of coordinates used to describe the spacetime. Our main goal today is to find this law.

3. Despite that your early experience w/ vectors may have been in terms of a ray connecting two positions (e.g. a position vector w.r.t. the origin), vectors do not in general live within a space or spacetime. This is because the usual rules of vector manipulation (linear algebra) require these objects to exist in flat spaces, i.e. $\mathbb{R}^n$ or $\mathbb{M}^n$. But spaces and spacetimes can be curved, e.g. S^2



How do we work with intrinsically flat objects in general curved spaces?
At each point in the space we define a <u>tangent space</u> as the set of tangent vectors to all curves passing through that point. Vectors at each point in the space live in these tangent spaces.
This has two important consequences:
a) We cannot freely move vectors around the space since in general the tangent spaces change.
b) Comparing vectors defined at two different points in a space will be tricky.
Understanding how to deal w/ these two consequences will require a lot of the technology that GR rests on.

But for now, let's stick to flat space. In this case all the tangent spaces are //.

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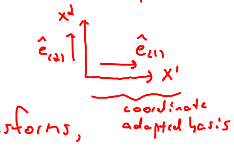
To uncover how vector components transform under coordinate changes, we first focus on a particularly simple vector ds. The components of this vector are just coordinate differentials dx^μ, so if we know how the coordinates themselves change, then we know how the components of <u>this</u> vector change. But we do:

$$x^\mu \to x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu \quad \Rightarrow \quad dx^\mu \to dx^{\mu'} = \Lambda^{\mu'}{}_\nu dx^\nu$$

Okay, so far we know how the components of one particular vector transform. But we would like a rule for the transformation of the components of any vector. To find this we will use a trick that we will also use when we extend our discussion to general tensors, so pay attention.

Recall that a vector is invariant under coordinate transformations. It's true that components change but the whole thing should not. How do we work with this?

Recall a vector is expressed as an expansion in some basis: $ds = \overbrace{dx^\mu}^{\text{components}} \underbrace{\hat{e}_{(\mu)}}_{\text{basis vectors}}$ ← labels vectors (not components)



The trick is that we want ds to be invariant, but then knowing how dx^μ transforms, we can get how $\hat{e}_{(\mu)}$ transforms.

$$ds = dx^\mu \hat{e}_{(\mu)} \to ds' = dx^{\mu'} \hat{e}_{(\mu')} = ds$$

$$= \Lambda^{\mu'}{}_\mu dx^\mu \hat{e}_{(\mu')} = \Lambda^{\mu'}{}_\mu dx^\mu \Lambda^\alpha{}_{\mu'} \hat{e}_{(\alpha)} = \underbrace{\Lambda^\alpha{}_{\mu'} \Lambda^{\mu'}{}_\mu dx^\mu \hat{e}_{(\alpha)} = dx^\mu \hat{e}_{(\mu)}}$$

guess $\hat{e}_{(\mu')} = \Lambda^\alpha{}_{\mu'} \hat{e}_{(\alpha)}$? WTF?!

If $\underbrace{\underbrace{\Lambda^\alpha{}_{\mu'}}_{\Lambda^{-1}} \underbrace{\Lambda^{\mu'}{}_\nu}_{\Lambda} = \underbrace{\delta^\alpha{}_\nu}_{I}}$

Remember: $\Lambda^T \eta \Lambda = \eta$
so $\Lambda^T \neq \Lambda^{-1}$!

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Don't get lost in all the index gymnastics. The end result should be straightforward. If ds is invariant, then the components and basis vectors should transform in ways that cancel each other.

In the end:

$$\boxed{\begin{aligned} & ds = dx^{\mu}\hat{e}_{(\mu)} \rightarrow dx^{\mu'}\hat{e}_{(\mu')} = ds' = ds \\ & \text{if } dx^{\mu} \rightarrow dx^{\mu'} = \Lambda^{\mu'}{}_{\nu}\, dx^{\nu} \\ & \qquad \hat{e}_{(\mu)} \rightarrow \hat{e}_{(\mu')} = \Lambda^{\alpha}{}_{\mu'}\, \hat{e}_{(\alpha)} \\ & \text{and } \Lambda^{\alpha}{}_{\mu'}\Lambda^{\mu'}{}_{\nu} = \delta^{\alpha}{}_{\nu} \end{aligned}}$$

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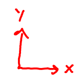
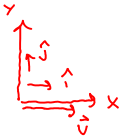
Now that we have the transformation law for the <u>basis vectors</u>, we recall that an <u>arbitrary vector</u> can be expanded as $V = V^\mu \hat{e}_{(\mu)}$ <u>and</u> should be invariant under coordinate changes. If you think about this for a few moments, hopefully you can see that this immediately tells us that the components of an arbitrary vector must transform like the components dx^μ! (If not, go back and review the last couple of pages carefully!)

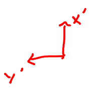
Thus: $$\boxed{\begin{aligned} &V = V^\mu \hat{e}_{(\mu)} \to V^{\mu'} \hat{e}_{(\mu')} = V' = V \\ &\text{if } V^{\mu'} = \Lambda^{\mu'}{}_\nu V^\nu \\ &\qquad \hat{e}_{(\mu')} = \Lambda^\alpha{}_{\mu'} \hat{e}_{(\alpha)} \\ &\text{and } \Lambda^\alpha{}_{\mu'} \Lambda^{\mu'}{}_\nu = \delta^\alpha{}_\nu \end{aligned}}$$ Transformation law for vector components and basis vectors.

Let's see this play out in a familiar example. Consider $\longrightarrow \vec{v}$ and its description in  and .

First: 

$$\vec{v} = v\hat{i} + 0\hat{j} = v^i \hat{e}_{(i)} \quad \text{where} \quad \hat{e}_{(1)} = \hat{i} = \begin{pmatrix}1\\0\end{pmatrix}, \quad \hat{e}_{(2)} = \hat{j} = \begin{pmatrix}0\\1\end{pmatrix}$$

$$v^1 = v, \; v^2 = 0$$

Now to get to  we rotate the <u>coordinates</u> w/ $R(90) = \begin{pmatrix}\cos 90 & \sin 90\\ -\sin 90 & \cos 90\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \Rightarrow R^{i'}{}_j$

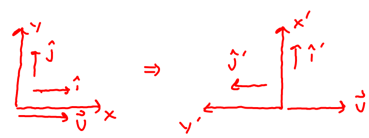
But to describe $\vec{v}$ in the new coordinates we first rotate the components by $R(90)$:

$$v^i \to v^{i'} = R^{i'}{}_j v^j \quad \Rightarrow \quad \begin{aligned} v^1 \to v^{1'} &= R^{1'}{}_1 v^1 + R^{1'}{}_2 v^2 = 0 \\ v^2 \to v^{2'} &= R^{2'}{}_1 v^1 + R^{2'}{}_2 v^2 = -v \end{aligned}$$

But then the basis vectors must be transformed by $R(90)^{-1} = R(-90) = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \Rightarrow R^j{}_{i'}$

$$\hat{e}_{(i)} \to \hat{e}_{(i')} = R^j{}_{i'} \hat{e}_{(j)} = R^1{}_{i'} \hat{e}_{(1)} + R^2{}_{i'} \hat{e}_{(2)} \quad \Rightarrow \quad \begin{aligned} \hat{e}_{(1')} &= R^1{}_{1'} \hat{e}_{(1)} + R^2{}_{1'} \hat{e}_{(2)} = 0\hat{e}_{(1)} + \hat{e}_{(2)} = \begin{pmatrix}0\\1\end{pmatrix} \\ \hat{e}_{(2')} &= R^1{}_{2'} \hat{e}_{(1)} + R^2{}_{2'} \hat{e}_{(2)} = -\hat{e}_{(1)} + 0\hat{e}_{(2)} = \begin{pmatrix}-1\\0\end{pmatrix} \end{aligned}$$

Comparing we see $\hat{i}' = \hat{j}$, $\hat{j}' = -\hat{i}$

Finally: $\vec{v} = v\hat{i} + 0\hat{j} = 0\hat{i}' - v\hat{j}'$ as expected from 

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Dual Vectors

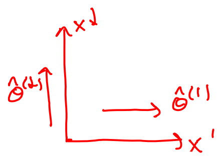
Believe it or not you have worked w/ dual vectors before (they are critical in defining a dot product). Chances are you didn't know because in $\mathbb{R}^n$ they look just like vectors.

We define dual vectors by 3 conditions:

1. Dual vectors are straight directed objects defined at a point in space (hence they must live in a copy of $\mathbb{R}^n$ or $\mathbb{M}^n$ at each point which we call the cotangent space).
2. Dual vectors are invariant. But given a coordinate system, they can be expressed in terms of components and dual basis vectors which do transform.
3. Dual vectors linearly eat vectors and poop scalars: $\underbrace{\omega}_{\text{dual vector}}(a\underbrace{V}_{\text{vector}} + b\underbrace{W}_{\text{vector}}) = a\,\omega(V) + b\,\omega(W) = \text{scalar}$

   scalars

Wait, if vectors and dual vectors are invariant, then what the hell is a scalar?
A scalar is an invariant whose explicit coordinate representation is <u>also</u> invariant.



Again we will use a coordinate adapted basis set of dual vectors $\hat{\theta}^{(\mu)}$.

Now for ordinary vectors we had the intuitive example $d\vec{s} = dx^\mu \hat{e}_{(\mu)}$ to start with. For dual vectors we don't have such a starting point. So we need to be clever ...

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Consider condition 3 for dual vectors. We will define "eating" in terms of the basis vectors and dual basis vectors as follows:

$$\underbrace{\hat{\theta}^{(\mu)}}_{\text{dual basis vector}} \underbrace{\hat{e}_{(\nu)}}_{\text{basis vector}} = \underbrace{\delta^{\mu}{}_{\nu}}_{\text{a number!}} = \begin{cases} +1 & \mu = \nu \\ 0 & \mu \neq \nu \end{cases}$$

Since $\delta^{\mu}{}_{\nu}$ is a number (scalar) it must be invariant, Hence:

$$\hat{\theta}^{(\mu)} \hat{e}_{(\nu)} = \delta^{\mu}{}_{\nu} \longrightarrow \hat{\theta}^{(\mu')} \underbrace{\hat{e}_{(\nu')}} = \delta^{\mu'}{}_{\nu'}$$

We already know: $\Lambda^{\nu}{}_{\nu'} \hat{e}_{(\nu)}$

So: $\hat{\theta}^{(\mu')} \Lambda^{\nu}{}_{\nu'} \hat{e}_{(\nu)}$

Guessing: $\Lambda^{\mu'}{}_{\mu} \hat{\theta}^{(\mu)} \Lambda^{\nu}{}_{\nu'} \hat{e}_{(\nu)} = \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu}{}_{\nu'} \underbrace{\hat{\theta}^{(\mu)} \hat{e}_{(\nu)}}_{\delta^{\mu}{}_{\nu}}$

$$= \underbrace{\Lambda^{\mu'}{}_{\mu}}_{\Lambda^{-1}} \underbrace{\Lambda^{\mu}{}_{\nu'}}_{\Lambda} = \delta^{\mu'}{}_{\nu'}$$

So we now know:

$$\boxed{\begin{aligned} \hat{e}_{(\mu)} &\rightarrow \hat{e}_{(\mu')} = \Lambda^{\mu}{}_{\mu'} \hat{e}_{(\mu)} \\ \hat{\theta}^{(\mu)} &\rightarrow \hat{\theta}^{(\mu')} = \Lambda^{\mu'}{}_{\mu} \hat{\theta}^{(\mu)} \end{aligned}}$$

Transformation laws for basis vectors and dual basis vectors.

But now that we know how dual basis vectors transform, we can use that an overall dual vector is invariant to determine how the components of a dual vector transform:

$$\boxed{\begin{aligned} V^{\mu} &\rightarrow V^{\mu'} = \Lambda^{\mu'}{}_{\mu} V^{\mu} \\ \omega_{\mu} &\rightarrow \omega_{\mu'} = \Lambda^{\mu}{}_{\mu'} \omega_{\mu} \end{aligned}}$$

Transformation law for components of vectors and dual vectors.

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We can now see the gorey details of dual vectors eating vectors:

$$\omega(V) = \omega_\mu \hat{\theta}^{(\mu)} V^\nu \hat{e}_{(\nu)}$$
$$= \omega_\mu V^\nu \delta^\mu{}_\nu$$
$$= \omega_\mu V^\mu = \underbrace{\omega_0 V^0 + \omega_1 V^1 + \omega_2 V^2 + \omega_3 V^3}_{} \in \mathbb{R}$$

This might look like a dot product between two vectors, but this is actually a combination of a vector and a dual vector!

So what is the dot product?

Recall that the dot product combines two vectors to make a number: $\bullet(V^\mu, V^\nu) \to \mathbb{R}$
In our language, what the dot product does is take one of the vectors and turn it into a dual vector, then let the dual vector eat the vector to make a number.
$\bullet(V^\mu, V^\nu) = V_\nu V^\nu \in \mathbb{R}$

But how to we take a given vector V^μ and create a corresponding dual vector V_μ?

We can take a hint from $\mathbb{R}^n$: $\bullet(V^i, V^j) = V^1 V^1 + V^2 V^2 + \cdots$
$$= \underbrace{\delta_{ij}}_{\text{metric on } \mathbb{R}^n} V^i V^j$$

So for SR, we replace $\delta_{ij} \to \eta_{\mu\nu}$
Hence: $\bullet(V^\mu, V^\nu) = \eta_{\mu\nu} V^\mu V^\nu = V_\nu V^\nu$

In general:
$$V^\mu \to V_\mu = \eta_{\mu\nu} V^\nu$$
$$V_\mu \to V^\mu = \underbrace{\eta^{\mu\nu}}_{\eta^{-1}} V_\nu$$
(note for SR $\eta^{-1} = \eta$, but this is not true in general!!)

So the metric (or its inverse) moves indices up or down (raise and lower) turning vectors into dual vectors and vice-versa.

Note: In all of this we are not changing coordinates!

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In the future we will largely drop all explicit reference to basis vectors $\hat{e}_{(\mu)}$ and dual basis vectors $\hat{\theta}^{(\mu)}$ since they take care of each other. We can instead just focus on the components.